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ANALYTICAL EXPRESSIONS FOR THE PHYSICAL CHARACTERISTICS NEAR INTERIOR POINTS OF POLYTROPES

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Abstract

In this paper, literal analytical expressions in power series forms are developed for the physical characteristics near interior points of polytropes.

Key words: Stellar interior, polytropes.

1. Introduction

Polytropic models are vital for two classes of theoretical astrophysics: stellar structure (Hunter, 2001; and Prialnik, 2007) and galactic dynamics (Binney and Tremains, 1987; and Bertin, 2000). In stellar structure one can obtain the march of the physical variables inside polytropic equilibrium configurations (Hansen et al., 2004). Most stellar models had more or less direct bearing on polytropes, and many analytical solutions to the internal constitution of stars are connected with polytropic relationships.

As, for examples, convective stellar models with radiation pressures and without radiation pressures where considered in full details long time ago. Öpik (1962) and Bobrov et al., (1978) have estimated the polytropic indices of planetary models obeying the polytropic equation of state. Magnetic braking of the rotation of pre-main sequence stars approximated as fully convective $n = 1.5$ polytropes, has been calculated by Okamoto (1969 and 1970). Vandervoort (1980) has considered an application to galactic bars of non-axisymmetric, triaxial polytropes with index $0.5 \le n \le 0.808$.

Moreover, particular models give systematic expositions of theoretical results, which bear some interest on the structure of rotating stars (Tassoul, 1978). White dwarfs and neutron stars are best studied using differentially rotating polytropic cores (Eriguchi and Mueller, 1985) Most stellar models had more or less direct bearing on polytropes, and many analytical solutions to the internal constitution of stars are connected with polytropic relationships. A model of pulsar that has a magnetized core was presented (Koichi, 2005) and the behaviour in evolution of spin and luminosity was analyzed. The stars are supposed to be polytropes, thereby rendering all physical quantities calculable with any desired precision. Recently

(Gonzalo, 2008), investigated spherically symmetric, static matter configurations with polytropic equation of state for a class of *f(R)* models in Palatini formalism.

In fact, the polytrope representation of stars models is a method that today still lends valuable technique and insights to the internal structure of stars. It is also proven to be most versatile in examination of a variety of situations, including the analysis of isothermal cores, convective stellar interiors and fully degenerate stellar configurations. Moreover, composite polytropes are used to construct spherical, hydrostatic models of molecular clouds (Charles and Christopher, 2000). Recently, Cristina (2007) obtained in the presence of a weak poloidal magnetic filed, the filed distribution in a sequence of composite polytrobic stars with uniform density throughout the core of polytropic index $n=0$, and an envelope with $n=1$.

On the other hand, in galactic dynamics, the Lane-Emden equation of the polytropic equilibrium configurations is considered as generating function of potential models for flattened galactic systems, upon such potentials the forces and the star orbits in these systems could be determined. These latter results are extremely important in galactic studies; one of their most explored aspects is the problem of the correlation between the parameters of the orbit of a star and its physical properties. Recently (Tung et al., 2007) used initial configurations with $n=3$ polytropes for axisymmetric simulations of the, magnetorotational collapse of very massive stars. Moreover, the effects of a positive cosmological constant on astrophysical and cosmological configurations described by polytrobic equation of state (Antolinez et al., 2007).

From the above brief notes, one can detected that, why a great effort has been devoted up to now, and is being devoted at present to express some characteristic physical parameters related to ploytrobic models (e.g. Sharaf et al., 1998; Sharaf et al., 2004 and Horedt, 2004).

The basic equation for ploytrobic configurations is the Lane-Emden differential equation.

In fact, in the absence of closed analytical solution of a given differential equation the power series solution (which of course assumed to be convergent) can serve as the analytical representation of its solution. Moreover, it is worth noting that the power series is one of the most powerful methods of mathematical analysis and is no less (and sometimes even more) convenient than the elementary functions especially when the problems are to be studied on computers. In fact, most computers often use series in the calculations of the majority of the elementary functions. Analytical solution for this equation was established (Sharaf and Sharaf, 1994) in a form of power series, which can be convergent for any allowable indices

n. Accelerated form of these power series are developed by Saad, (2004) using Pade' technique and changing of the independent variable. For polytropic and isothermal gas spheres (N=3), Nouh, (2004) improved the convergence of the power series by using a combination of two accelerating techniques Eular-Abel transformation and Pade' approximation.

The physical characteristics of polytropic configurations depends in turn on the solutions of Lane-Emden equation and have been widely quoted in the astrophysical literatures.

Due to the importance of ploytrobic configurations as mentioned in brief, and the needs of having numerical and analytical treatments for their structures so as to suit any application are what motivated the present paper with the following objective:

"To establish literal analytical expressions in power series forms for the physical characteristics of the ploytrobic configurations."

The importance of these expressions is due to some factors of these are:

(i) Their analytical forms, offer in general much deeper insight into the nature of the physical characteristics to which they refer.

(ii) These expressions are general in the sense that they could be used for any polytrobic index n, so they can suit many of the applications.

(iii) The coefficients of each of these expressions have been found directly without the use of recurrence formulae.

2. Lane-Emden equation and its analytical solution:

Consider equations of state of the form

$$P = K\,\rho^{\gamma}\,, \tag{1}$$

where K and γ are constants, known as polytropic equations of state. It is customary to define the corresponding polytropic index, denoted by n, as

$$\gamma = 1 + \frac{1}{n}\,. \tag{2}$$

Let r the space coordinate giving the distance from the centre of the star. Consequently, the density and the pressure are functions of the distance r.

Lane-Emden equation of index n is given by [e.g., Prialnik, 2007]

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n\,, \tag{3}$$

where the dimensionless variable ξ, is defined by

$$\alpha = \left(\frac{(n+1)\;K}{4\pi\,G}\,\rho_c^{\frac{1}{n}-1}\right)^{1/2}\;; \qquad r = \alpha\,\xi$$

ρ_c the central density and G is the gravitational constant its numerical value is

$$G = 6.673 \times 10^{-8}\ \text{cm}^3/\text{gm}/\text{sec}^2.$$

The relation between ρ and θ is given as

$$\rho = \rho_c\,\theta^n\,. \tag{4}$$

Equation (3) is to be solved under the conditions

$$\text{at} \quad \xi = 0; \quad \theta = 1; \quad \frac{d\theta}{d\xi} = 0\,. \tag{5}$$

With these initial conditions, the second order differential Equation (3) (Lane-Emden equation) will possess a unique solution. This solution is called the Lane-Emden function of index n and is denoted by $\theta = \theta_n(\xi)$.

2.1 $\forall\, 0 \le \xi \le 1$ Power series solution of Lane-Emden equation

The power series solution of Lane-Emden equation valid $\forall\ 0 \le \xi \le 1$ is (Sharaf and Sharaf, 1994) given as

$$\theta_n(\xi) = 1 + \sum_{k=1}^{\infty} a_k\,\xi^{2k}\,, \tag{6}$$

$$a_1 = -1/6\,, \tag{7}$$

which satisfies the initial conditions of Equation (5) , and the *a's* coefficients are completely determined in full recursive way from Equation (7) and

$$a_{k+1} = \frac{1}{k(k+1)(2k+3)}\sum_{i=1}^{k}(ni-k+i)(2k-2i+3)(k+1-i)\,a_i\,a_{k+1-i} \quad \forall\ k\ge 1. \tag{8}$$

The well known closed analytical solution of Lane-Emden Equation for $n = 0, 1$ and 5 could easily obtained from Equations (6) and (8), and we get

$$\left.\begin{aligned}\theta_0(\xi) &= 1 - \xi^2/6\\ \theta_1(\xi) &= (\sin\xi)/\xi\\ \theta_5(\xi) &= \left(1+\xi^2/3\right)^{-1/2}\end{aligned}\right\}. \tag{9}$$

2.2 Symbolic expressions of the a's coefficients

Due to the limited space, the symbolic expressions of the *a's* coefficients are listed in Table (A1) of the Appendix for j = 1, 2, …, 7.

3. Symbolic Expressions of the physical Characteristics of polytropes:

In this section literal analytical expressions in power series forms are established for the physical characteristics of polytropes. These series converge very rabidly

$\forall\ 0 \le \xi \le 1$. Due to the limited space, only the first five symbolic expressions of the coefficients for each physical characteristic will be listed in a separate table.

3.1 The Mass

The mass interior to ξ is given as

$$M(\xi) = 4\pi\alpha^3 \rho_c \left(-\xi^2 \frac{d\theta}{d\xi} \right),$$

then using Equation (6) we get

$$M(\xi) = \sum_{j=1}^{\infty} \lambda_j \, \xi^{2j+1} \ ,$$

$$\lambda_j = -8\pi \ j \alpha^3 \ \rho_c \, a_j \, .$$

Symbolic expressions of the coefficients λ_j ; j = 1, 2, 3, 4, 5 are listed in Table (A2) of the Appendix.

3.2 The Mass-radius relation

The mass-radius relation is given as

$$\frac{M(\xi)}{r(\xi)} = \frac{4\pi\alpha^3 \rho_c \left(-\xi^2 \, d\theta / d\xi \right)}{\alpha \, \xi} \Rightarrow \frac{M(\xi)}{r(\xi)} = -4\pi\alpha^2 \rho_c \, \xi \ d\theta / d\xi \ .$$

Using Equation (6) we get

$$\frac{M(\xi)}{r(\xi)} = \sum_{j=1}^{\infty} v_j \, \xi^{2j} \ ,$$

and $$v_j = -8\pi \ j \alpha^2 \ \rho_c \, a_j \, .$$

Symbolic expressions of the coefficients v_j ; j = 1, 2, 3, 4, 5 are listed in Table (A3) of the Appendix.

3.3 Central condensation

The central condensation is given as

$$\frac{\rho_c}{\overline{\rho}} = -\frac{3}{\xi} \left(d\theta / d\xi \right)^{-1},$$

where $\overline{\rho}$ is the mean density. Using Equation (6) we get

$$\frac{\rho_c}{\overline{\rho}} = \sum_{j=0}^{\infty} \eta_j \, \xi^{2j} \ ,$$

$$\eta_0 = 1 \, ,$$

$$\eta_m = \frac{2}{3}\sum_{k=1}^{m}(k+1)\,a_{k+1}\,\eta_{m-k} \quad \forall\, m \geq 1.$$

Symbolic expressions of the coefficients η_j ; j = 1, 2, 3, 4, 5 are listed in Table (A4) of the Appendix.

3.4 The Temperature

The temperature is given as

$$T = T_c\,\theta\,.$$

Using Equation (6), we get

$$T = \sum_{j=0}^{\infty}\gamma_j\,\xi^{2j}\,,$$

$$\gamma_0 = T_c\,,$$

$$\gamma_j = a_j\,T_c\,.$$

Symbolic expressions of the coefficients γ_j ; j = 1, 2, 3, 4, 5 are listed in Table (A5) of the Appendix.

3.5 Pressure

The pressure is given as

$$P(\xi) = P_c\,\theta^{n+1}.$$

Using Equation (6), we get

$$P(\xi) = \sum_{j=0}^{\infty}\beta_j\,\xi^{2j}\,, \qquad \beta_0 = P_c\,,$$

$$\beta_m = \frac{1}{m}\sum_{k=1}^{m}(n\,k + 2k - m)\,a_k\,\beta_{m-k} \quad \forall\, m \geq 1.$$

Symbolic expressions of the coefficients β_j ; j = 1, 2, 3, 4, 5 are listed in Table (A6) of the Appendix.

3.6 The Density

Since

$$\rho = \rho_c\,\theta^n\,.$$

Using Equation (6), we get

$$\rho = \sum_{j=0}^{\infty}\chi_j\,\xi^{2j}\,, \qquad \chi_0 = \rho_c\,,$$

$$\chi_m = \frac{1}{m}\sum_{k=1}^{m}(n\,k + k - m)\,a_k\,\chi_{m-k} \quad \forall\, m \geq 1.$$

Symbolic expressions of the coefficients χ_j ; j = 1, 2, 3, 4, 5 are listed in Table (A7) of the Appendix.

3.7 Gravity

The gravity $g(\xi)$ could be written as

$$g(\xi) = -4\pi G \left\{ \frac{(n+1)k}{4\pi G} \right\}^{1/2} \rho_c^{\frac{n+1}{2n}} \frac{d\theta}{d\xi} .$$

Then using Equation (6), we get

$$g(\xi) = \sum_{j=1}^{\infty} \delta_j \, \xi^{2j-1} ,$$

$$\delta_j = -8\pi \, j G \left\{ \frac{(n+1)k}{4\pi G} \right\}^{1/2} \rho_c^{\frac{n+1}{2n}} a_j .$$

Symbolic expressions of the coefficients δ_j ; j = 1, 2, 3, 4, 5 are listed in Table (A8) of the Appendix.

3.8 Gravitational acceleration

The gravitational acceleration is given by

$$F = -G \frac{M(\xi)}{r^2(\xi)} = -G \frac{4\pi\alpha^3 \rho_c \left(-\xi^2 \, d\theta / d\xi \right)}{\alpha^2 \, \xi^2} = -4\pi \, \alpha \, G \, \rho_c \, d\theta / d\xi .$$

Using Equation (6), we get

$$F = \sum_{j=1}^{\infty} \tau_j \, \xi^{2j-1} ,$$

$$\tau_j = 8\pi \, j \, \alpha \, G \, \rho_c \, a_j .$$

Symbolic expressions of the coefficients τ_j ; j = 1, 2, 3, 4, 5 are listed in Table (A9) of the Appendix.

4. Conclusion

In concluding the present paper, literal analytical expressions in power series forms are developed for the physical characteristics near interior points of polytropes.

The importance of these expressions are due to some facts, of these are the following:

1. The expressions are obtained as power series in ξ. Consequently, we can obtain physical characteristics *X* (say) very simply and efficiently by using any power series evaluation algorithms. On the other hand, the solution of Lane-Emden equation to obtain the physical characteristics *X* by any numerical differential equation solver gives us *X* only at definite values of ξ belonging to the set *S*, where

$$S = \{0, \Delta\xi, 2\Delta\xi, 3\Delta\xi, \ldots\}$$

where $\Delta\xi$ the step size used in numerical differential equation solver. So if we need the values of X at $\xi^* \notin S$, we must apply an interpolation formula. A process, which needs more execution time, moreover, which is the most critical, the loss of accuracy that usually associated with the usage of interpolation formula.

2. The analytical power series representations of the physical characteristics are invariant under many operations because, addition, multiplication, exponent ion, integration, different ion, etc of a power series is also a power series. A fact which provides excellent flexibility in dealing with analytical as well as computational developments of the problems of polytropes.

Once more, numerical differential equation solver can not, by any way, provide such flexibility. Moreover, these analytical formulae usually offer much deeper insight into the nature of a physical characteristic as compared to numerical integration.

Appendices

Table (A1): Symbolic expressions of the coefficients a_j ; j = 1, 2, 3, 4, 5, 6, 7.

$$a_1 = \frac{1}{6},$$

$$a_2 = \frac{n}{120},$$

$$a_3 = \frac{(5-8n)n}{15120},$$

$$a_4 = \frac{n[70+61n(-3+2n)]}{3265920},$$

$$a_5 = \frac{n[-3150+n\{10805+2n(-6321+2516n)\}]}{1796256000},$$

$$a_6 = \frac{n[138600+n\{-574850+n(915935+2n(-331583+91808n))\}]}{840647808000},$$

$$a_7 = \frac{n\begin{bmatrix}-21021000+n\{101038350+ \\ n(-199037015+2n(100286893+4n\{-12897299+2703152n\}))\}\end{bmatrix}}{1235752277760000}.$$

Table (A2): Symbolic expressions of the coefficients λ_j ; j = 1, 2, 3, 4, 5.

$$\lambda_1 = \frac{4}{3}\pi\,\alpha^3\,\rho_c,$$

$$\lambda_2 = -\frac{2}{15}n\,\pi\,\alpha^3\,\rho_c,$$

$$\lambda_3 = \frac{1}{630} n(-5+8n)\pi\alpha^3 \rho_c \,,$$

$$\lambda_4 = -\frac{n\{70+61n(-3+2n)\}\pi\alpha^3 \rho_c}{102\,060} \,,$$

$$\lambda_5 = \frac{n\{-3150+n[10805+2n(-6321+2516n)]\}\pi\alpha^3 \rho_c}{44\,906\,400} \,.$$

Table (A3): Symbolic expressions of the coefficients ν_j ; j = 1, 2, 3, 4, 5.

$$\nu_1 = \frac{4}{3}\pi\alpha^2 \rho_c \,,$$

$$\nu_2 = -\frac{2}{15} n\pi\alpha^2 \rho_c \,,$$

$$\nu_3 = \frac{1}{630} n(-5+8n)\pi\alpha^2 \rho_c \,,$$

$$\nu_4 = -\frac{n[70+61n(-3+2n)]\pi\alpha^2 \rho_c}{102\,060} \,,$$

$$\nu_5 = \frac{n[-3150+n\{10805+2n(-6321+2516n)\}]\pi\alpha^2 \rho_c}{44\,906\,400} \,.$$

Table (A4): Symbolic expressions of the coefficients η_j ; j = 1, 2, 3, 4, 5.

$$\eta_0 = 1 \,,$$

$$\eta_1 = \frac{n}{90} \,,$$

$$\eta_2 = \frac{n(75-106n)}{113\,400} \,,$$

$$\eta_3 = \frac{n[1750+n(-4125+2372n)]}{30\,618\,000} \,,$$

$$\eta_4 = -\frac{n[4961250+n\{15568625+26n(-619725+208304n)\}]}{848\,730\,960\,000} \,,$$

$$\eta_5 = \frac{n\begin{bmatrix} 218295000 + \\ n\{-837366250 + n(1201480125 + 2n(-378096625 + 86712968\, n))\} \end{bmatrix}}{331\,005\,074\,400\,000}$$

.

Table (A5): Symbolic expressions of the coefficients γ_j ; j = 1, 2, 3, 4, 5.

$$\gamma_0 = T_c\,,$$

$$\gamma_1 = -\frac{T_c}{6}\,,$$

$$\gamma_2 = \frac{n\,T_c}{120}\,,$$

$$\gamma_3 = -\frac{n(-5+8n)T_c}{15120}\,,$$

$$\gamma_4 = \frac{n[70+61n(-3+2n)]T_c}{3\,265\,920}\,,$$

$$\gamma_5 = -\frac{n[-3150+n\{10805+2n(-6321+2516n)\}]T_c}{1796\,256\,000}\,.$$

Table (A6): Symbolic expressions of the coefficients β_j ; j = 1, 2, 3, 4, 5.

$$\beta_0 = P_c\,,$$

$$\beta_1 = -\frac{1}{6}(1+n)\,P_c\,,$$

$$\beta_2 = \frac{1}{45}n(1+n)\,P_c\,,$$

$$\beta_3 = -\frac{n(1+n)(-25+61n)\,P_c}{22680}\,,$$

$$\beta_4 = \frac{n(1+n)[175+n(-660+629n)]\,P_c}{2\,041\,200}\,,$$

$$\beta_5 = -\frac{n(1+n)[-11025+n\{52310+n(-84309+45904n)\}]\,P_c}{1347\,192\,000}\,.$$

Table (A7): Symbolic expressions of the coefficients χ_j ; j = 1, 2, 3, 4, 5.

$$\chi_0 = \rho_c\,,$$

$$\chi_1 = -\frac{n\,\rho_c}{6}\,,$$

$$\chi_2 = \frac{n(-5+8n)\,\rho_c}{360}\,,$$

$$\chi_3 = -\frac{n[70+61n(-3+2n)]\,\rho_c}{45\,360}\,,$$

$$\chi_4 = \frac{n[-3150 + n\{10805 + 2n(-6321 + 2516n)\}]\rho_c}{16329600},$$

$$\chi_5 = -\frac{n\begin{bmatrix}138600+ \\ n\{-574850 + n[915935 + 2n(-331583 + 91808n)]\}\rho_c\end{bmatrix}}{5388768000}.$$

Table (A8): Symbolic expressions of the coefficients δ_j ; j = 1, 2, 3, 4, 5.

$$\delta_1 = \frac{2}{3}G\sqrt{\frac{R(1+n)}{G}}\sqrt{\pi}\,\rho_c^{\frac{1+n}{2n}},$$

$$\delta_2 = -\frac{1}{15}G\,n\sqrt{\frac{R(1+n)}{G}}\sqrt{\pi}\,\rho_c^{\frac{1+n}{2n}},$$

$$\delta_3 = \frac{1}{1260}G\,n\sqrt{\frac{R(1+n)}{G}}(-5+8n)\sqrt{\pi}\,\rho_c^{\frac{1+n}{2n}},$$

$$\delta_4 = -\frac{G\,n\sqrt{\frac{R(1+n)}{G}}[70 + 61n(-3+2n)]\sqrt{\pi}\,\rho_c^{\frac{1+n}{2n}}}{204120},$$

$$\delta_5 = \frac{G\,n\sqrt{\frac{R(1+n)}{G}}[-3150 + n\{10805 + 2n(-6321 + 2516n)\}]\sqrt{\pi}\,\rho_c^{\frac{1+n}{2n}}}{89812800}.$$

Table (A9): Symbolic expressions of the coefficients τ_j ; j = 1, 2, 3, 4, 5.

$$\tau_1 = -\frac{4}{3}G\,\pi\,\alpha\,\rho_c,$$

$$\tau_2 = \frac{2}{15}G\,n\,\pi\,\alpha\,\rho_c,$$

$$\tau_3 = -\frac{1}{630}G\,n(-5+8n)\,\pi\,\alpha\,\rho_c,$$

$$\tau_4 = \frac{G\,n[70 + 61n(-3+2n)]\pi\,\alpha\,\rho_c}{102060},$$

$$\tau_5 = -\frac{G\,n[-3150 + n\{10805 + 2n(-6321 + 2516n)\}]\pi\,\alpha\,\rho_c}{44906400}.$$

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