STUDYING STABILITY OF THE LIBERATION POINTS OF BINARY ASTEROIDS

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ABSTRACT

In this study, the locations of the equilibrium points of both triangular and collinear of restricted three-body problem and their stabilities are studied. This study was applied on ten binary asteroids. A code was constructed by MATHEMATICA language to obtain liberation points and their stabilities.

On the other hand, the contour of zero velocity curves was displayed for two stable and unstable binaries.

Key words: binary asteroids, liberation points, stability.

1. INTRODUCTION


In this study, the locations of the collinear and triangular points of ten binary asteroids have been computed and their stabilities have been determined.

2. Equation Of Motion Of The Restricted Three-Body Problem

The restricted three-body problem refers to the dynamics of two bodies of masses m₁ ≤ m₂ (referred to as the primaries) that move along circles about their common center of mass, and of a third body, of infinitesimal mass, that is subject to the gravitational attraction of the primaries. The motion of the primaries is not affected by the motion of the infinitesimal mass. Fig.(2.1) illustrates the position of the third body m₃ referring to the center of mass of m₁ and m₂, and the reference plane (x, y, z).

Figure 2.1 inertial frame of three bodies.

The equations of motion for third body in synodic barycentric coordinate are given by (Szebehely, 1967):

$$\ddot{x} - 2\dot{y} = x - \frac{\mu_1}{r_1^3}(x - x_1) - \frac{\mu_2}{r_2^3}(x - x_2),$$

(2.1)
where
\[ \mu = \frac{m_3}{m_1 + m_2} : \text{mass ratio} \]
\[ r_1^2 = (x - x_1)^2 + y^2 + z^2 : \text{distance from } m_1 \text{ to } m_3, \]
\[ r_2^2 = (x - x_2)^2 + y^2 + z^2 : \text{distance from } m_2 \text{ to } m_3, \]
\[ \mu_1 = \frac{Gm_1}{r_1^3} = 1 - \mu : \text{gravitational parameter for } m_1, \]
\[ \mu_2 = \frac{Gm_2}{r_2^3} = \mu : \text{gravitational parameter for } m_2, \]
\[ x_1 = -\mu_2 = -\mu : \text{distance from } m_1 \text{ to mass center}, \]
\[ x_2 = \mu_1 = 1 - \mu : \text{distance from } m_2 \text{ to mass center}. \]

3. Liberation points

At the liberation points there are zero velocity regions. So that it is very important to specify the location of these points.

Figure (3.1) shows the liberation points (L₁, L₂, L₃, L₄, and L₅) for the two primary bodies (m₁ and m₂).

Figure 3.1: The five Lagrange Points associated with two primary bodies.

Some restrictions are considered to determine the locations of the liberation points, which are:

1. m₃ lies at any point of (L₁, L₂, L₃, L₄, L₅).
2. m₃ is very smaller than m₁, m₂.
3. the third mass would have zero velocity and zero acceleration where would appear permanently at rest relative to m₁ and m₂ and the equilibrium points are defined when
   \[ \ddot{x} = \ddot{y} = \ddot{z} = 0, \]
   \[ \dddot{x} = \dddot{y} = \dddot{z} = 0. \]

Substituting Eqs. (3.1) and (3.2) into Eqs. (2.1), (2.2) and (2.3) respectively, this yields
\[ x = \frac{\mu_1}{r_1^3}(x + \mu) + \frac{\mu_2}{r_2^3}(x - (1 - \mu)), \]
\[ y = \left[ \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} \right] y, \]
\[ \left[ \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} \right] z = 0. \]

3.1. Location of liberation points of L₄ and L₅

After some little algebraic calculations had been done to solve (3.3), (3.4) and (3.5) then it is found that
\[ x = \frac{1}{2} - \mu, \]
\[ y = \pm \frac{\sqrt{3}}{2}. \]

So, the coordinates of L₄ and L₅ are being

L₄ \((\frac{1}{2} - \mu, \frac{\sqrt{3}}{2})\) and
L₅ \((\frac{1}{2} - \mu, -\frac{\sqrt{3}}{2})\).

3.2. Location of liberation points of L₁, L₂, and L₃

L₁, L₂ and L₃

Recall Eqs. (2.1), (2.2) and (2.3) with y = 0 as well as z = 0, then the three collinear points
\((L₁, L₂, L₃)\) could be found from Eq. (3.3)

Now we can calculate \((L₄, L₅, L₆)\) from
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1. For \( L_1 \) lies between masses \( m_1, m_3 \) and \( m_2 \) it can be calculated from nonlinear equation

\[
\frac{x - \left( \frac{1 - \mu}{x + \mu} \right)}{(x + \mu)^2} + \frac{\mu}{(x - (1 - \mu))^2} = 0.
\]  

2. For \( L_2 \) lies outside the mass \( m_2 \) and it can be calculated from nonlinear equation

\[
\frac{x - \left( \frac{1 - \mu}{x + \mu} \right)}{(x + \mu)^2} - \frac{\mu}{(x - (1 - \mu))^2} = 0.
\]

3. For \( L_3 \) point lies on the negative \( x \)-axis and it can be calculated from nonlinear equation

\[
x + \left( \frac{1 - \mu}{x + \mu} \right) \frac{\mu}{(x - (1 - \mu))^2} = 0.
\]

4. Jacobi Integral

Multiply Eq. (2.1) by \( \dot{x} \), Eq. (2.2) by \( \dot{y} \) and Eq. (2.3) by \( \dot{z} \) to obtain

\[
\ddot{x} + 2\dot{x}\dot{y} - \ddot{y} = -\frac{\mu_1}{r_1^3} \dot{x} + \frac{\mu_2}{r_2^3} \dot{x} - \frac{\mu_3}{r_3^3} \dot{y},
\]

\[
\ddot{y} + 2\dot{x}\dot{y} - \ddot{y} = -\frac{\mu_1}{r_1^3} \dot{y} - \frac{\mu_2}{r_2^3} \dot{y} - \frac{\mu_3}{r_3^3} \dot{y},
\]

\[
\ddot{z} = -\frac{\mu_1}{r_1^3} \dot{z} - \frac{\mu_2}{r_2^3} \dot{z} - \frac{\mu_3}{r_3^3} \dot{z}.
\]

After some algebraic calculations, the integration was done to obtain the zero velocity curves (Moulton, 1970)

\[
\frac{1}{2} \dot{x}^2 - \frac{1}{2} \left( x^2 + y^2 \right) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} = c
\]

where

\[
\frac{1}{2} \dot{v}^2 : \text{is kinetic energy per unit mass relative to the rotating frame,}
\]

\[-\frac{\mu_1}{r_1} \text{ and } -\frac{\mu_2}{r_2} : \text{are the gravitational potential energy of the two primary masses respectively,}
\]

\[c : \text{is called Jacobi integral, or Jacobi constant,}
\]

\[\text{sometimes called the integral of relative energy.}
\]

Equations (4.1) illustrate the zero velocity curves when \( v = 0 \).

5. Stability of the liberation points

5.1 - Firstly at collinear points

To study the motion near any of the equilibrium point \( L(x_0, y_0) \)

\[
x = x_0 + \xi y
\]

\[
y = y_0 + \eta
\]

where \( \xi \) and \( \eta \) are the coordinate (5.2) and the potential \( V \)

\[
V = \frac{1}{2} (x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}
\]

So, \( V \) may be expanded by Taylor series \( \epsilon \)

\[
v = V(x_0, y_0) + V_x(x_0, y_0) \xi + V_y(x_0, y_0) \eta + \frac{1}{2} V_{xx}(x_0, y_0) \xi^2 + V_{xy}(x_0, y_0) \xi \eta
\]

\[
+ \frac{1}{2} V_{yy}(x_0, y_0) \eta^2
\]

where

\[V_x \text{ is first derivative of } V \text{ with respect to } x,
\]

\[V_y \text{ is first derivative of } V \text{ with respect to } y,
\]

\[V_{xx} \text{ is second derivative of } V \text{ with respect to } x,
\]

\[V_{yy} \text{ is second derivative of } V \text{ with respect to } y,
\]

The equation of motion of three body could be written in suitable form as

\[
\ddot{x} - 2\dot{y} = V_x,
\]

\[
\ddot{y} - 2\dot{x} = V_y,
\]

\[
\ddot{z} = V_z.
\]

Substitute from Eqs. (5.1) and (5.2) into Eqs. (5.4), (5.5) and (5.6) respectively then it is found that

\[
\dot{\xi} - 2\dot{\eta} = \xi V_{xx} + V_{xy} \eta,
\]

\[
\dot{\eta} + 2\dot{\xi} = \xi V_{xy} + V_{yy} \eta,
\]

\[
\dot{\zeta} = \xi V_{zz}.
\]

Let

\[
\xi = \alpha e^{\lambda t},
\]

\[
\eta = \beta e^{\lambda t}.
\]

Substitute from Eqs. (5.10) and (5.11) into Eqs (5.7), (5.8) and (5.9) respectively then it is found that

\[
(\lambda^2 - V_{xx}) \alpha = (2\lambda + V_{xy}) \beta,
\]

\[
(\lambda^2 - V_{xy}) \beta = (2\lambda + V_{yy}) \alpha.
\]

\[
(\lambda^2 - V_{yy}) \alpha = (2\lambda + V_{xx}) \beta.
\]
The characteristic equation becomes (Moulton, 1970):

\[(2\lambda - V_{xy})\alpha = (\lambda^2 - V_{yy}) \xi, \]  \hspace{1cm} (5.13)

By solving eq. (5.14) the roots of \(\lambda\) are obtained. If the roots obtained are pure imaginary numbers, then \(\xi\) and \(\eta\) are periodic and this stable periodic solution in the vicinity of \(x_0\) and \(y_0\) can be studied as:

1. If any of the \(\lambda\) roots are real or complex numbers, then \(\xi\) and \(\eta\) increase with time so that the solution is unstable. This can be happened because the solution contains constants terms in the form of exponentials.

2. If the remaining exponentials are purely imaginaries. Then the solution is stable.

To obtain the expressions \(V_{xx}, V_{yy}, V_{xy}\), \(V_{xz}\) and \(V_{yz}\) in terms of \(r_1, r_2\) and \(\mu\) let

\[r_i^2 = (x - x_i)^2 + y^2 + z^2, \quad i = 1, 2; \]

\[A = \frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3}, \]

\[B = 3 \left( \frac{1 - \mu}{r_1^5} + \frac{\mu}{r_2^5} \right), \]

\[C = 3 \left( \frac{1 - \mu}{r_1^7} (x_0 - x_1) + \frac{\mu}{r_2^7} (x_0 - x_2) \right). \]

In the case of collinear points, \(y_0 = z_0 = 0\), so that

\[r_i^2 = (x_0 - x_i)^2, \quad i.e., \quad i = 1, 2 \]  \hspace{1cm} (5.18)

\[V_{xy} = V_{xz} = V_{yz} = 0. \]

Then, the equations of motion become

\[\ddot{\xi} - 2\dot{\eta} = \xi V_{xx}; \quad \ddot{\eta} + 2\dot{\xi} = \eta (1 + A), \]  \hspace{1cm} (5.19)

\[\ddot{\xi} = \eta V_{yy}; \quad \ddot{\xi} = -A \xi. \]  \hspace{1cm} (5.20)

The Last equation is independent of the first two Eqs. and its solution is

\[\xi = c_1 \sin t + c_2 \cos t. \]

Therefore the motion parallel to the z-axis for small all displacement is periodic with period \(2\pi \sqrt{\mu}A\). Applying the values of \(V_{xx}, V_{xy}\) and \(V_{yy}\), \(V_{zz}\) in equation (5.14) yields

\[\lambda^2 + (2 - A)\lambda^2 + (1 + A - 2A^2) = 0 \]  \hspace{1cm} (5.22)

Now there are three values of \(A\) corresponding to the three Lagrangian points \(L_1, L_2, L_3\) obtained from equations (3.2.1), (3.2.2) and (3.2.3) respectively. It can be shown that the values of \(L_1, L_2, L_3\) the next condition is verified.

\[1 + A - 2A^2 < 0. \]

While values of \(\mu\) up to its limit 1/2. Then, the four roots of equation (5.22) consist of two real roots, numerically equal but opposite in sign, and two conjugate pure imaginary roots. Then the solution for the straight-line case is unstable and the orbit becomes spiral.

5.2-Secondly triangular points:

The coordinate of the triangular equilibrium points \(L_4, L_5\) are

\[x_0 = \frac{1}{2} - \mu, \quad y_0 = \pm \frac{\sqrt{3}}{2} \]

At \(L_4\) from Eqs (5.15), (5.16), (5.17) and (5.22) it is found that

\[V_{xx} = \frac{3}{4} V_{yy} = \frac{9}{4} V_{zz} = -1. \]  \hspace{1cm} (5.15)

\[V_{xz} = -2 \sqrt{3}, \quad V_{yz} = \frac{3}{4} (1 - 2\mu), \quad V_{xy} = \frac{3}{4} \sqrt{3}. \]  \hspace{1cm} (5.23)

The equations of motion at \(L_4\) become
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The last equation is independent of the first two and its solution is
\[ \zeta = c_1 \sin t + c_2 \cos t. \]
so that the motion parallel to the z-axis for small displacement and the solution is periodic with period \( 2\pi. \)

As the same way to determine the characteristic equation for collinear points it is found that the characteristic equation for \( L_4 \) becomes
\[ \lambda^4 + \lambda^2 + \frac{27}{4} \mu(1 - 2\mu) = 0. \quad (5.26) \]

If \( \mu \leq \frac{1}{2} \) and if \( 1 - 27 \mu(1 - 2\mu) \geq 0 \) the roots are pure imaginary.

The inequality may be written as
\[ 1 - 27 \mu(1 - 2\mu) = \varepsilon, \]
where \( \varepsilon \) is a positive quantity whose limit is zero.

The solution of this equation is
\[ \mu = \frac{1}{2} \pm \sqrt{\frac{23 + 4\varepsilon}{108}}. \]

Since \( \mu \) represents the mass ratio, which is less than \( \frac{1}{2} \) the negative sign must be taken at the limit, \( \varepsilon = 0 \rightarrow \mu = 0.0385, 0. \)

Therefore if \( \mu < 0.0385 \) the roots become pure imaginaries and the motion of the particle displaced from the equilibrium point is oscillatory in form, hence the particle will remain in the vicinity of equilibrium point and the motion become stable.

If \( \mu > 0.0385 \) the roots become complex and the orbits become spiral.

The spiral orbits asymptotically approach the triangular libration points or depart from them; therefore, the motion becomes unstable.

6. RESULTS AND CONCLUSION

A code was constructed by MATHEMATICA language, and applied on the ten binary asteroids to obtain liberation points, mass ratio and drawing contour plot of zero velocity curves. Selected two binary asteroids One of them its triangular points stable and another unstable at \( L_4 \) and \( L_5 \). By using these relations to determine mass ratio from half-diameter of each mass binary asteroids as
\[ \mu = \frac{m_2}{m_1 + m_2} = \frac{\frac{4}{3} \pi \rho R^3}{\frac{4}{3} \pi \rho (R^3_1 + R^3_2)} = \frac{R^3_2}{R^3_1 + R^3_2}. \]

Tables (6.1) and (6.2) show the results for the given data of two asteroid diameters (D1 and D2) and the distance between them (\( R_b \)), which are used by the code to determine the liberation points \( (L_1, L_2, L_3, L_4, \text{and} L_5) \) for the ten binary asteroids. Figures (6.1) and (6.2) display the contours of zero velocity curves for stable binary 1996FG3 at \( \mu = 0.02829 \) and for unstable binary 1999 DJ4 at \( \mu = 0.1111 \) respectively.
Table 6.1: collinear points and stability for binary system.

<table>
<thead>
<tr>
<th>No</th>
<th>Binary</th>
<th>$D_1$ (km)</th>
<th>$D_2$ (km)</th>
<th>$R_2$ (km)</th>
<th>$\mu$</th>
<th>$L_1$ (km)</th>
<th>$L_2$ (km)</th>
<th>$L_3$ (km)</th>
<th>Stability For $L_1, L_2, L_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>VH 1991</td>
<td>1.1</td>
<td>44.</td>
<td>3.2</td>
<td>0.0601</td>
<td>2.212</td>
<td>3.959</td>
<td>-3.28</td>
<td>unstable</td>
</tr>
<tr>
<td>2</td>
<td>AW1 1994</td>
<td>1</td>
<td>49.</td>
<td>2.3</td>
<td>1.0526</td>
<td>1.337</td>
<td>2.901</td>
<td>-2.40</td>
<td>unstable</td>
</tr>
<tr>
<td>3</td>
<td>FG3 1996</td>
<td>1.5</td>
<td>465.</td>
<td>2.6</td>
<td>0.02892</td>
<td>2.009</td>
<td>3.118</td>
<td>-2.63</td>
<td>unstable</td>
</tr>
<tr>
<td>4</td>
<td>PG 1998</td>
<td>9.</td>
<td>27.</td>
<td>1.5</td>
<td>0.02629</td>
<td>1.127</td>
<td>1.79</td>
<td>-1.51</td>
<td>unstable</td>
</tr>
<tr>
<td>5</td>
<td>DJ4 1999</td>
<td>35.</td>
<td>175.</td>
<td>8.</td>
<td>11111.</td>
<td>47.</td>
<td>9093.</td>
<td>836.</td>
<td>unstable</td>
</tr>
<tr>
<td>6</td>
<td>KW4 1999</td>
<td>1.5</td>
<td>57.</td>
<td>2.54</td>
<td>0.05201</td>
<td>1.804</td>
<td>3.124</td>
<td>-5.595</td>
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<td>DP107 2000</td>
<td>8.</td>
<td>328.</td>
<td>2.6</td>
<td>0.06447</td>
<td>1.77</td>
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<tr>
<td>8</td>
<td>UG11 2000</td>
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<td>156.</td>
<td>4.</td>
<td>17763.</td>
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<td>508.</td>
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<tr>
<td>9</td>
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<td>8.</td>
<td>224.</td>
<td>1.4</td>
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<tr>
<td>10</td>
<td>CE26 2002</td>
<td>3</td>
<td>21.</td>
<td>5.1</td>
<td>0.0034.</td>
<td>4.854</td>
<td>5.349</td>
<td>-5.1007</td>
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</table>

Table 6.2: triangular points and stabilities for binary system.

<table>
<thead>
<tr>
<th>No</th>
<th>Binary</th>
<th>$L_4$ (x, y)</th>
<th>$L_5$ (x, y)</th>
<th>Stability For $L_4, L_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>VH 1991</td>
<td>1.407</td>
<td>2.771</td>
<td>Unstable</td>
</tr>
<tr>
<td>2</td>
<td>AW1 1994</td>
<td>0.908</td>
<td>1.991</td>
<td>Unstable</td>
</tr>
<tr>
<td>3</td>
<td>FG3 1996</td>
<td>1.285</td>
<td>2.251</td>
<td>Stable</td>
</tr>
<tr>
<td>4</td>
<td>PG 1998</td>
<td>0.711</td>
<td>1.299</td>
<td>Stable</td>
</tr>
<tr>
<td>5</td>
<td>DJ4 1999</td>
<td>0.311</td>
<td>0.692</td>
<td>Unstable</td>
</tr>
<tr>
<td>6</td>
<td>KW4 1999</td>
<td>1.139</td>
<td>2.199</td>
<td>Unstable</td>
</tr>
<tr>
<td>7</td>
<td>DP107 2000</td>
<td>1.132</td>
<td>2.251</td>
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</tr>
<tr>
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<td>0.346</td>
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<tr>
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<td>SL9 2001</td>
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<td>1.212</td>
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</tr>
<tr>
<td>10</td>
<td>CE26 2002</td>
<td>2.584</td>
<td>4.42</td>
<td>Stable</td>
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</tbody>
</table>
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Figure 6.1: Contour plot for 1996 FG3 binary system with $\mu = 0.02829$

Figure 6.2: Contour plot for 1999 DJ4 binary system with $\mu = 0.1111$
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REFERENCES:
Gabern, F. and Jorba, A; 1991. “Restricted four and five body problem in the solar system” universitat de Barcelona gran via 585,08007 Barcelona, Spain.
Inga Jinang and Lia-chin Yeh.;2004."The modified restricted three body problem”. The environment of evolution of couple and multiple stars .IA,UNAM.
Euler, Leonhard, 1773, De moturectilineotriumcorporum se mutuoattrahentium (http://www.math.dartmouth.edu/~euler/docs/originals/E327.pdf)
Sharma, R. K.; 1980."Periodic orbits of the second kind in the restricted three body problem when the more massive primary”. VirkarmSarabahaiSpace Center, Trivandrum, India.
Sharivastava. A.K. and Ishwar B.;1983. “Equation of motion of the restricted problem of three bodies with variable mass”. Department of mathematics .india college engineering, Bihar, India