HALLEY'S FUNCTION FOR REAL POLYNOMIALS IS INCREASING

OMAR ISMAEL EL HASADI

Department of Astronomy, Faculty of Science, University Omar Al Mukhtar, Branch Derna, Libya,

Abstract. In this paper we want to show that Halley's function for real polynomial is an increasing rational homeomorphism map on $\mathbb{R}$.

Keywords: Halley's function, Derivative of Halley's method, homeomorphism, Increasing Function.

Introduction

Halley's method is an elegant method for finding roots and a third-order algorithm. Such an algorithm converges cubically insofar as the number of significant digits eventually triples with each iteration. And not only does the first derivative of a third-order iteration vanish at a fixed point, but so does the second derivative. In this paper, we recall some definitions, theorems for Halley's function for a real polynomial and the derivative of Halley's function. Then we conclude that Halley's function for real polynomial is an increasing rational homeomorphism map on $\mathbb{R}$.

0.1 Halley's method for real polynomials

In this section, our objective is to study the iteration of Halley's function associated with a polynomial $p$ of degree $d$ with real coefficients and only real (and simple) zeros $x_k$, $1 \leq k \leq d$. This method is equivalent to iterating the rational map

$$H_p(z) = z - \frac{2p'(z)p(z)}{2(p'(z))^2 - p(z)p''(z)},$$

(0.1.1)

where

$$p(z) = a_0 + a_1z + a_2z^2 + \ldots + a_{d-1}z^{d-1} + a_dz^d.$$

So if $p(z)$ has degree $d$ and has distinct roots, then by a simple calculation $H_p(z)$ is a rational map of degree $2d - 1$. As for the case of Newton's method, the roots of $p(z)$ are fixed points of $H_p(z)$, although other fixed points exist as well. Since we are assuming that the roots of $p(z)$ are distinct, the critical points of $p(z)$ are also fixed points under Halley's method.

E-mail: allhasady@yahoo.com
0.2 Derivative of Halley’s method

The derivative of Halley’s method is

\[ H_p'(z) = - \frac{(p(z))^3 S[p](z)}{2 \left( p'(z) - \frac{p(z)p''(z)}{2p'(z)} \right)^2}, \tag{0.2.1} \]

where \( S[p](z) \) is the Schwarzian derivative of \( p(z) \), that is

\[ S[p](z) = \frac{p'''(z)}{p'(z)} - 3 \left( \frac{p''(z)}{p'(z)} \right)^2. \tag{0.2.2} \]

From expression (0.2.1), we can see that the roots are super-attracting fixed points, but of one degree higher order than for Newton’s method.

As we know that the degree of Halley’s function is \( 2d - 1 \), where \( d \) is the degree of the polynomial \( p \), there are \( 4d - 4 \) critical points, \( 2d \) of them coincide with the roots \( x_k \), and \( 2d - 4 \) are free critical points placed at points where the Schwarzian derivative of \( p(z) \) vanishes.

**Remark 0.2.1.** The second derivative of \( H_p \) vanishes at \( x_k \), whereas as the second derivative of \( N_p \) does not, the graph of \( H_p \) is flatter than that of \( N_p \) near the fixed point. This accounts for the difference in speed at which the two algorithms converge (see [6], [3] for details). In general, the higher the order, the flatter the graph, the faster convergence.

**Theorem 0.2.1.** Let

\[ H_p(z) = z - \frac{2p'(z)p(z)}{2(p'(z))^2 - p(z)p''(z)}, \]

where \( p \) is a polynomial with real (and simply) distinct zeros. Then \( H_p \) has no real pole.

**Proof.** We will show that

\[ (p')^2 - pp'' > 0 \quad \text{on} \quad \mathbb{R}, \]

which is known as Polya’s result.

Write

\[ (p')^2 - pp'' = p^2 \left( \frac{(p')^2}{p^2} - \frac{p''}{p} \right) = p^2 \left( \left( \frac{p'}{p} \right)^2 - \frac{p''}{p} \right). \]

We know that

\[ \left( \frac{p'}{p} \right)^2 = \left( \sum_{j=1}^{d} \frac{1}{z - x_j} \right)^2, \]
where $x_j$ are roots of $p$, $1 \leq j \leq d$, hence

$$
\frac{p''}{p} = \left( \sum_{j=1}^{d} \frac{1}{z - x_j} \right)^2 - \sum_{j=1}^{d} \frac{1}{(z - x_j)^2}.
$$

From

$$
\sum_{j=1}^{d} \frac{1}{(z - x_j)^2} > 0, \quad z \in \mathbb{R},
$$

it follows that

$$
\left( \frac{p'}{p} \right)^2 > \frac{p''}{p},
$$

hence

$$
2(p')^2 - pp'' > 0.
$$

Thus $H_p$ does not have any real pole.
Theorem 0.2.2. Let $H_p(z)$ be a Halley’s function for a polynomial $p(z)$, then $H_p'(z) \geq 0$ on $\mathbb{R}$.

Proof. We know that

$$H_p'(z) = -\frac{(p'(z))^2 S[p](z)}{2 \left( p'(z) - \frac{p(z) p''(z)}{2p'(z)} \right)^3},$$

where $S[p](z)$ is the Schwarzian derivative of $p(z)$, that is

$$S[p](z) = \frac{p'''(z)}{p'(z)} - \frac{3}{2} \left( \frac{p''(z)}{p'(z)} \right)^2 = \frac{2p''p'' - 3p'''}{2p'(z)^2}. $$

To show that $H_p'(z) \geq 0$, we have to prove that $S[p](z) < 0$. By the same proof as before, we can see that $(p''')^2 - 3p''' > 0$, then $2p''p''' - 3(p''')^2 < 0$. Thus $S[p](z) < 0$, implies $H_p'(z) \geq 0$.

Conclusion 1. From theorems (0.2.1) and (0.2.2), we conclude that $H_p$ is an increasing rational homeomorphism map on $\mathbb{R}$.

References